CHAPTER 1

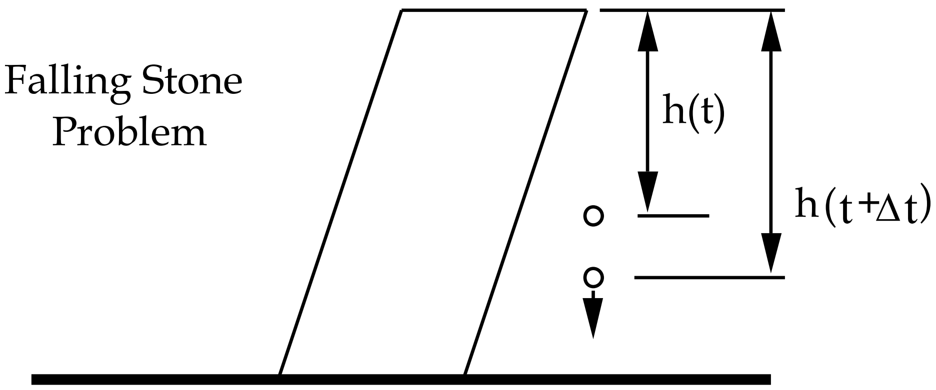
FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

1 Introduction

1.1 Applications of ODEs - A Simple Example

Physical Problem

$\Rightarrow$ Math Modeling
$\Rightarrow$ Solving the Math Problem
$\Rightarrow$ Interpretation of Its Physical Meaning



$$F = ma = mg \quad \text{where} \quad a = g = \text{acceleration of gravity}$$

$$\frac{h(t+\Delta t) - h(t)}{\Delta t} = v(t) \text{ or, for } \Delta t \to 0, \quad \boxed{\frac{dh}{dt} = v}$$

$$\text{Since} \quad a = \frac{dv}{dt} = \frac{d^2h}{dt^2}$$

$$\text{we have} \quad \boxed{\frac{d^2h}{dt^2} = a = g} \approx \text{constant}$$

Math Model!

The above equation can be written as

$$\frac{d}{dt}\left[\frac{dh}{dt}\right] = g$$

$\boxed{\text{Mathematical Modelling}}$

$$\frac{dh}{dt} = g\,t + c_1 \qquad \text{or} \qquad v = g\,t + c_1$$

$$\text{And} \quad h = \frac{1}{2}\,g\,t^2 + c_1\,t + c_2$$

I.C. (Initial Condition):

$$h(0) = 0 \qquad \Rightarrow \qquad c_2 = 0$$
$$v(0) = 0 \qquad \Rightarrow \qquad c_1 = 0$$

Thus, we have $\quad h = \frac{1}{2}\,g\,t^2$ #

$\boxed{\text{Solution of the Math Problem}}$

h = falling distance
g = acceleration of gravity
t = time

$\boxed{\text{Interpretation of Physical Meaning}}$

A Physical Problem

$\Rightarrow$ Mathematical Model

$\Rightarrow$ **Solution of the Mathematical Problem**

$\Rightarrow$ Interpretation in Terms of Its Physical Meaning

1.2 Some Definitions

(1) Ordinary Differential Equation - *only one independent variable*

$$y = y(x)$$

y: dependent variable; x: independent variable

$$f(y, x, y', y'', \ldots) = 0$$

where $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}$

Partial Differential Equation - *more than one independent variable!*

$$\varphi = \varphi(x, t, \cdots)$$

$$\Phi\left(\varphi, x, t, \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial t}, \frac{\partial^2\varphi}{\partial x^2}, \frac{\partial^2\varphi}{\partial t^2}, \frac{\partial^2\varphi}{\partial x\partial t}, \cdots\right) = 0$$

where

$\varphi =$ dependent vatiable

$x,\ t\ =$ independent variables

(2) Order: *the highest derivative* of y with respect to x in the equation

$$y'' + 4y' + 5y = 0 \qquad 2^{nd} \text{ order}$$
$$y' - y\cos x = 0 \qquad 1^{st} \text{ order}$$
$$(y^{(4)})^{3/5} - 2y'' = \cos x \qquad 4^{th} \text{ order}$$

A first order ODE can be written either in an *implicit form* or an *explicit form*:

$$F(x, y, y') = 0 \quad \text{or} \quad y' = f(x, y)$$

(3) Solution:

A **solution** of a given 1^{st}-order differential equation on some open interval $a < x < b$ is a function $y = h(x)$ that *has a derivative* $y' = h'(x)$ and *satisfies this equation for all x in that interval*; that is, the equation becomes an identity if we replace the unknown function y by h and y′ by h′.

A **solution** of an n^{th}–order differential equation is a function that is *n times differentiable* and that *satisfies the differential equation*.

(a) General solution: contains arbitrary constants, e.g.,

$$\frac{d^2h}{dt^2} = g$$

$$\Rightarrow \quad h(t) = \frac{1}{2} g t^2 + c_1 t + c_2$$

where c_1 and c_2 are constants and are arbitrary.

(b) Particular solution: no arbitrary constants

$$\frac{d^2h}{dt^2} = g$$

with $\frac{dh}{dt}(t=0) = 0, \; h(t=0) = 0$ (initial conditions)

$$\Rightarrow \quad h(t) = \frac{1}{2} g t^2$$

(c) Trivial solution: If $y = 0$ is a solution to a differential equation on an interval I, then $y = 0$ is called the trivial solution to that differential equation on I. e.g.,

$$\frac{dy}{dx} = 3y$$

$y = 0$ —— trivial solution

$y = c\, e^{3x}$ —— general solution

(d) Explicit solution: $y = f(x)$

e.g., $y = c\, e^{3x}$ is an explicit solution of $y' = 3y$

(e) Implicit solution: $f(x, y) = 0$

e.g., $x^2 + y^2 - 1 = 0$ is an implicit solution of $y\, y' = -x$.

(f) Singular solution: a solution can't be obtained from the general solution

e.g., $y'^2 - x\,y' + y = 0$

$y = c\,x - c^2$ —— general solution

$y = x^2/4$ —— singular solution

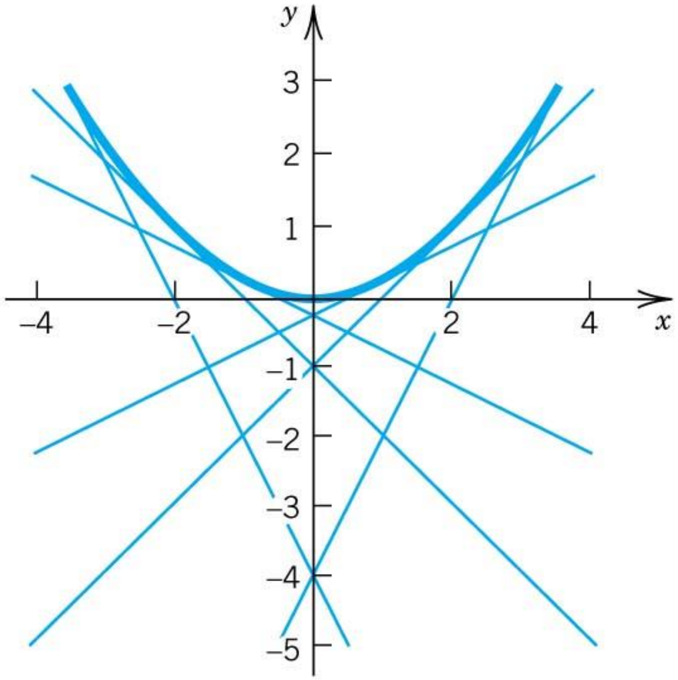


Fig. 2. Singular solution (parabola) and particular solutions of (5)

(4) Verification of Solution

[Example] The solution of $xy' = 2y$ is $y = x^2$.

Verify: $x(2x) = 2x^2 = 2y$.

[Example] The solution of $yy' = -x$ on the interval $-1 < x < +1$ is $x^2 + y^2 - 1 = 0 \ (y > 0)$.

Verify: $2x + 2yy' = 0 \Rightarrow yy' = -x$.

- **There are equations that do not have solutions at all. For example,** $(y')^2 = -1$ **does not have a real-valued solution.**

- **There are equations that do not have general solutions. For example,** $|y'| + |y| = 0$ **has only a trivial solution** $y = 0$**.**

2 Separable Differential Equations

2.1 Separation of Variables

If the differential equation can be reduced to the form $\boxed{y' = \dfrac{f(x)}{g(y)}}$

$$g(y)\, y' = f(x)$$

or, since $\quad y' = \dfrac{dy}{dx}$

$$g(y)\, dy = f(x)\, dx$$

then we have a ***separable* equation** and the general solution can be obtained by integration on both sides:

$$\boxed{\int g(y)\, dy = \int f(x)\, dx + \mathbf{c}}$$

where c is an arbitrary constant.

[Example] $$\frac{dy}{dx} = x\sqrt{1-y^2}$$

[Solution] $$\frac{dy}{\sqrt{1-y^2}} = x\,dx$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int x dx + c$$

or $$\sin^{-1} y = \frac{x^2}{2} + c \quad \text{implicit solution}$$

or $$y = \sin\left(\frac{x^2}{2} + c\right) \text{ explicit solution}$$

[Example]

$$y' = ky$$

[Solution]

$$\frac{dy}{y} = kdx \qquad \Rightarrow \qquad \ln|y| = kx + \tilde{c}$$

$$\left(\begin{array}{ll} y > 0 & \qquad (\ln|y|)' = (\ln y)' = y'/y \\ y < 0 & (\ln|y|)' = (\ln(-y))' = -y'/-y = y'/y \end{array}\right)$$

$$|y| = e^{\tilde{c}} e^{kx} \qquad \Rightarrow \qquad y = ce^{kx}$$

where

$$c = \begin{cases} +e^{\tilde{c}} & y > 0 \\ -e^{\tilde{c}} & y < 0 \\ 0 & y = 0 \end{cases}$$

Trivial solution!

[Example] $y' = -2xy$

[Solution] $$\frac{dy}{dx} = -2xy \rightarrow \frac{dy}{y} = -2x\,dx$$

$$\therefore \quad \ln|y| = -x^2 + c$$

or $$|y| = \exp\{-x^2 + c\} \quad \text{or} \quad y = c' e^{-x^2}$$

Note that if $c' = 0$, we have a trivial solution $y = 0$.

[Example]

$$y' = -y / x \qquad\qquad y(1) = 1$$

[Solution]

$$\frac{dy}{y} = -\frac{dx}{x} \qquad\qquad \ln|y| = -\ln|x| + \tilde{c} \qquad \therefore\ y = \frac{c}{x}$$

where

$$c = \begin{cases} +e^{\tilde{c}} & x, y > 0 \quad \text{or} \quad x, y < 0 \\ -e^{\tilde{c}} & x > 0, y < 0 \quad \text{or} \quad x < 0, y > 0 \end{cases}$$

From IC: $y(1) = c = 1 \qquad \Rightarrow \qquad xy = 1$

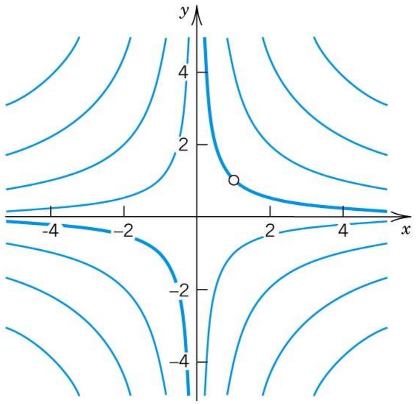


Fig. 4. Solutions of $y' = -y/x$ (hyperbolas)

[Exercises] Please solve the following equations

(i) $e^{x-y}\dfrac{dy}{dx} + 1 = 0$

(ii) $y' = 2xy \quad ; \quad y(0) = 1$

(iii) $y' = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

2.2 Initial Value Problems

Ordinary Differential Equation + Initial Condition(s)

$$y' = f(x, y) \qquad\qquad y(x_0) = y_0$$

[Example] $(x^2+1)y' + (y^2+1) = 0 \quad y(0) = 1$

[Solution] $$\frac{dy}{y^2+1} = -\frac{dx}{x^2+1}$$

$$\tan^{-1}y = -\tan^{-1}x + c$$

or $$\tan^{-1}x + \tan^{-1}y = c$$

$$\tan(\tan^{-1}x + \tan^{-1}y) = \tan c$$

or $$\frac{x+y}{1-xy} = \tan c = c' \text{ —— } \textbf{general solution}$$

Since $y(0) = 1$, we have $\tan c = 1$

$\Rightarrow$ $$\frac{x+y}{1-xy} = 1$$

or $$y = \frac{1-x}{1+x} \quad \# \qquad \text{—— } \textbf{particular solution}$$

Note that in general there is no arbitrary constant for initial value problems.

3 Equations Reducible to Separable Forms

(1) $y' = f(ax + by + c)$ where a, b and c are constants

Let $u = ax + by + c$

$$\frac{du}{dx} = a + b\frac{dy}{dx} = a + b\,f(u)$$

$$\Rightarrow \quad \int \frac{du}{a + b\,f(u)} = \int dx + c$$

[Example] $y' = (x+y)^2 + a^2$ $\quad a$ = constant

Let $\quad u = x + y \quad \therefore y' = u^2 + a^2$

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \Rightarrow \quad \frac{du}{dx} = 1 + u^2 + a^2$$

$$\therefore \frac{du}{u^2 + a^2 + 1} = dx \quad \text{and} \quad \int \frac{du}{u^2 + a^2 + 1} = x + c$$

$$\frac{1}{\sqrt{1+a^2}} \tan^{-1} \frac{u}{\sqrt{1+a^2}} = x + c$$

$$u = \sqrt{1+a^2} \tan\left(x\sqrt{1+a^2} + c' \right) \quad \text{where} \quad c' = c\sqrt{1+a^2}$$

$$\therefore \quad y = \sqrt{1+a^2} \tan\left(x\sqrt{1+a^2} + c' \right) - x \ \#$$

(2) $y' = f(y/x)$

Let $y/x = u$ or $y = x\,u$

$$\frac{dy}{dx} = \boxed{x\frac{du}{dx} + u = f(u)}$$

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

$$\int \frac{du}{f(u) - u} = \ln|x| + c$$

or $$x = c_1 \exp\left\{\int \frac{du}{f(u) - u}\right\}$$

[Example] $y' = \dfrac{y - kx}{x + ky}$ where $k = \text{constant}$

Let $y/x = u$ or $y = x\,u$

$\therefore$ $y' = u + x\,u'$

$$u + x\,u' = \frac{u - k}{1 + ku} \; ; \quad \text{or} \quad x\,u' = -\frac{k\,(1 + u^2)}{1 + ku}$$

$$\Rightarrow \quad \left\{ \frac{1}{k(1 + u^2)} + \frac{u}{1 + u^2} \right\} du + \frac{dx}{x} = 0$$

$$\therefore \quad \frac{1}{k} \tan^{-1} u + \ln \sqrt{1+u^2} + \ln |x| = c$$

$$\text{or} \quad \frac{1}{k} \tan^{-1} \frac{y}{x} + \ln \sqrt{x^2+y^2} = c \quad \#$$

[Exercises] Please solve the following equations:

$$\text{(i)} \quad y' = \frac{xy + 2y^2}{x^2}$$

$$\text{(ii)} \quad y' = \frac{y - x}{y + x}$$

$$\text{(iii)} \quad y' = \frac{2x^{-1}y - 3}{2y^{-1}x - 3}$$

(3) $y' = f\left(\dfrac{Ax + By + C}{ax + by + c}\right)$**, where A, B, C, a, b, c, are constants**

a. $C = c = 0$

$$y' = f\left(\frac{Ax + By}{ax + by}\right) = f\left(\frac{A + B(y/x)}{a + b(y/x)}\right) = g(y/x)$$

$\Rightarrow$ Same as Case (2)

b. $C \neq 0$ **and/or** $c \neq 0$

(i) $A\,b - B\,a \neq 0$

(ii) $A\,b - B\,a = 0$ or $\dfrac{a}{A} = \dfrac{b}{B}$

b. $C \neq 0$ and/or $c \neq 0$: (i) $Ab - Ba \neq 0$

Let $x = t + h$

$y = z + d$

where h and d are *constants to be determined* later

$$\frac{dy}{dx} = \frac{dz}{dt} = f\left(\frac{At + Bz + Ah + Bd + C}{at + bz + ah + bd + c}\right)$$

We may take appropriate h and d such that

$$\begin{cases} Ah + Bd + C = 0 \\ ah + bd + c = 0 \end{cases}$$

then, the differential equation reduces to

$$\frac{dz}{dt} = f\left(\frac{At + Bz}{at + bz}\right)$$

$\Rightarrow$ Same as Case (3)a or Case (2)

b. $\quad$ **$C \neq 0$ and/or $c \neq 0$:** (ii) $A\,b - B\,a = 0$ or $\dfrac{a}{A} = \dfrac{b}{B}$

(a) $a = b = 0$

$$y' = f\left(\frac{Ax + By + C}{c}\right) = F\,(px + qy + r)$$

$\Rightarrow$ Same as Case (1)

(b) Set

$A\,x + B\,y + C = u$ or $a\,x + b\,y + c = u$

$\Rightarrow$ Same as Case (1)

[Example] $(7y - 3x + 3)\, y' + 3y - 7x + 7 = 0$

[Solution] The above ODE can be written as

$$y' = \frac{7x - 3y - 7}{-3x + 7y + 3}$$

Check $\quad A\,b - B\,a = 7\times 7 - (-3)\times(-3) \neq 0$

Let $x = t + h$ and $y = z + d$ and take h and d such that

$$\begin{cases} Ah + Bd + C = 0 \\ ah + bd + c = 0 \end{cases} \quad \text{or} \quad \begin{cases} 7h - 3d - 7 = 0 \\ -3h + 7d + 3 = 0 \end{cases}$$

$$\therefore \quad d = 0 \qquad h = 1 \qquad \text{and} \qquad t = x - 1 \qquad z = y$$

$$\therefore \quad \frac{dy}{dx} = \frac{dz}{dt} = \frac{7(x-1) - 3y}{-3(x-1) + 7y} = \frac{7t - 3z}{-3t + 7z}$$

thus $\dfrac{dz}{dt} = \dfrac{7t - 3z}{-3t + 7z} \Rightarrow$ Case (2)

Ans: $\quad (y + x - 1)^5 (y - x + 1)^2 = C \qquad$ Please check it !!

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[Example] $(y - x + 5)\, y' = y - x + 1$

[Solution] $A\,b - B\,a = 0$

Let $u = y - x + 1 \quad \therefore y' = \dfrac{u}{u+4}$

$$\frac{du}{dx} = \frac{dy}{dx} - 1 = y' - 1 \qquad \therefore \quad y' = 1 + u'$$

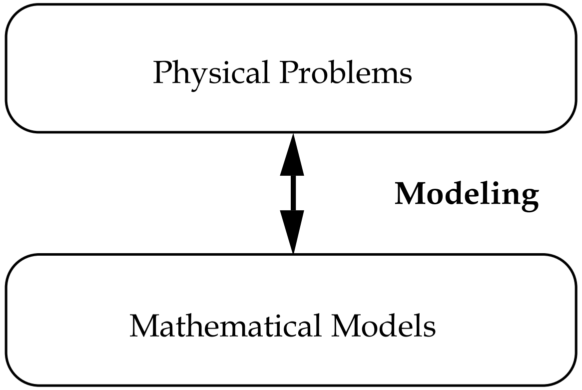
$$\Rightarrow \quad (u + 4)(1 + u') = u \qquad \text{or} \qquad (u + 4)\, u' = -4$$

$$\int (u + 4)\, du = \int -4\, dx + c$$

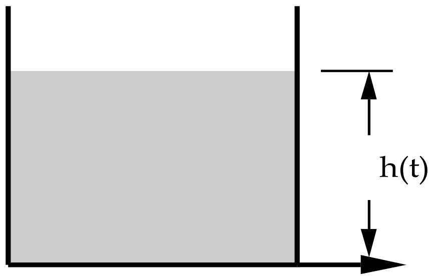
$$\therefore \quad \frac{1}{2} u^2 + 4u + 4x = c \qquad \text{i.e.,} \quad (y - x)^2 + 10\, y - 2\, x = c'$$

[Exercise] Solve $y' = \dfrac{1 - 2y - 4x}{1 + y + 2x}$

4 Modeling



[Example] Time required for draining a tank –– Torricelli's Law



$\Delta V = A\ v\ \Delta t$

where ΔV = volume of water flows out during Δt

A = cross-sectional area of the outlet = 0.7854 cm^2

v = velocity of the out-flowing water

Torricelli's law states that

$v = 0.6\sqrt{2gh}$

where g = acceleration of gravity = 980 cm/sec^2

h = height of water above the outlet

Note that ΔV must equal to change of the volumes of water in the tank, i.e.,

$\Delta V = -\,B\,\Delta h$ (Mass Balance)

where B = cross-sectional area of the tank = 7854 cm^2

Δh = decrease of the height h(y) of the water

i.e., $A\,v\,\Delta t = -\,B\,\Delta h$

or
$$\frac{\Delta h}{\Delta t} = -\frac{Av}{B} = -\frac{A\,0.6\sqrt{2gh}}{B}$$

Letting $\Delta t \to 0$, we obtain the differential equation

$$\frac{dh}{dt} = -\frac{A\,0.6\sqrt{2gh}}{B} = -\,0.00266\sqrt{h}$$

Initially, the height of the water is 150 cm, i.e.,

$$h(0) = 150 \text{ cm} \qquad \text{—— Initial Condition}$$

we then have

$$h(t) = (12.25 - 0.00133\,t)^2 \qquad \#$$

5 Exact Differential Equations

5.1 Total Differential (Exact Differential)

The total differential du of a function of two variables u(x,y) is defined by

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

e.g., if u(x,y) = x y, then the total differential of u is

$$du = y\, dx + x\, dy$$

Suppose that we take the total differential of the equation u(x,y) = c, then

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0$$

e.g., the total differential of the equation

$$x\, y = c$$

is $\quad y\, dx + x\, dy = 0 \quad$ or $\quad y' = -y/x \quad$ **(ODE!)**

Reversing the situation, suppose that we start with the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If we can find a function u(x,y) such that

$$\frac{\partial u}{\partial x} = M(x,y) \qquad \frac{\partial u}{\partial y} = N(x,y)$$

then the differential equation becomes

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

which has the general solution

$$u(x,y) = c$$

In this case, the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad \left(\text{or } \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x,y) \right)$$

is called an ***exact differential equation***.

5.2 Condition for Exact Differential

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential equation, then

$$M(x,y) = \frac{\partial u}{\partial x} \quad ; \quad N(x,y) = \frac{\partial u}{\partial y}$$

But
$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Thus,
$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

is the ***necessary and sufficient condition*** for $Mdx + Ndy$ to be a total differential.

5.3 Solution for Exact Differential Equations

Method I

Since $$\frac{\partial u}{\partial x} = M(x,y)$$

the solution has the following form

$$u = \int M\,dx + k(y) = c$$

To determine k(y), we take $\frac{\partial u}{\partial y}$ of the above equation and compare the result with

$$\frac{\partial u}{\partial y} = N(x,y)$$

Method II

$$u = \int N dy + l(x) = c'$$

$$\frac{\partial u}{\partial x} = M(x, y)$$

[Example] Solve $x\,y' + y + 4 = 0$

[Solution] $(y + 4)\,dx + x\,dy = 0$

or $M = y + 4 \quad ; \quad N = x$

Check the exactness by

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x} \Rightarrow \text{Exact Differential}$$

Solve for $u = \int M\,dx + k(y) = \int (y + 4)\,dx + k(y) = x\,y + 4x + k(y)$

But $\dfrac{\partial u}{\partial y} = N(x,y)$ or $x + k'(y) = x$

$\therefore\ k'(y) = 0$ or $k(y) = c^*$. Thus, we have the solution $u = c$.

$$\therefore \quad x\,y + 4x + c^* = c \quad \text{or} \quad x\,y + 4x = c' \quad \#$$

Differentiate wrt x, i.e. $\dfrac{du}{dx} = y + xy' + 4 = 0 \ \therefore y' = -\dfrac{y+4}{x} = -\dfrac{M}{N}$

[Example] $(1 - \sin x \tan y)\, dx + (\cos x \sec^2 y)\, dy = 0$

[Solution] $M = 1 - \sin x \tan y$

$N = \cos x \sec^2 y$

$$\frac{\partial M}{\partial y} = -\sin x \sec^2 y = \frac{\partial N}{\partial x} \quad \Rightarrow \quad \text{Exact}$$

Differential

The solution is

$$u = \int M\, dx + k(y)$$

$$= \int (1 - \sin x \tan y)\, dx + k(y) = x + \cos x \tan y + k(y)$$

$$\frac{\partial u}{\partial y} = N(x,y) \rightarrow \cos x \sec^2 y + k(y)' = \cos x \sec^2 y$$

$\therefore \quad k'(y) = 0$ or $k(y) = c^*$

The solution u = c becomes: $x + \cos x \tan y = c'$ #

6 Integrating Factors

A Simple Example:

$$\frac{1}{y}\,dx + 2\,x\,dy = 0$$

Since $\frac{\partial M}{\partial y} = -\frac{1}{y^2} \neq 2 = \frac{\partial N}{\partial x}$, it is not exact!

Multiply both sides of the above equation by F(x,y)=y/x (integrating factor), then

$$\frac{y}{x}\left(\frac{1}{y}\right)dx + \frac{y}{x}(2x)\,dy = 0$$

$$\frac{dx}{x} + 2\,y\,dy = 0$$

Since $\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$, it is exact!

A differential equation which is not exact can be made exact by multiply it by a suitable function F(x,y) ($\neq$ 0). This function is then called an ***integrating factor*.**

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \text{—— not exact}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$$F(x,y)\,P(x,y)\,dx + F(x,y)\,Q(x,y)\,dy = 0 \qquad \text{—— Exact}$$

In this case, we need to solve

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

which is a **partial differential equation** of F(x,y). In general, it is difficult to determine an integrating factor from the above equation. However, in some special cases, the integrating factor can be found as shown in the following *special cases*:

(i) **If $\dfrac{\left[\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right]}{Q}=f(x)$, i.e., a function of x only, then $F(x)=e^{\int f(x)dx}$ is an integrating factor, which is also a function of x only.**

[Proof]

Since the integrating factor satisfies the PDE

$$\frac{\partial}{\partial y}(FP)=\frac{\partial}{\partial x}(FQ)$$

$$\therefore F\frac{\partial P}{\partial y}+P\frac{\partial F}{\partial y}=F\frac{\partial Q}{\partial x}+Q\frac{\partial F}{\partial x} \quad \textbf{or} \quad F\left\{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right\}=Q\frac{\partial F}{\partial x}-P\frac{\partial F}{\partial y}$$

If we assume that the integrating factor F is function of x only, i.e.,

$$F=F(x) \qquad \Rightarrow \qquad \frac{\partial F}{\partial y}=0$$

i.e., $$F\left\{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right\}=Q\frac{dF}{dx} \quad \textbf{or} \quad \frac{d\ln F}{dx}=\frac{\left\{\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right\}}{Q}=f(x)$$

The above equation can be solved if the right-hand-side is function of x only, i.e., $\dfrac{d\ln F}{dx}=f(x)$, and the solution is then the integrating factor

$$F=e^{\int f(x)dx}$$

(ii) If $\dfrac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{P} = f(y)$, i.e., a function of y only, then $e^{-\int f(y)dy}$ is an integrating factor, which is also a function of y only.

[Proof]: Exercise!

(iii) If $\dfrac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q - P} = f(x+y) = f(v)$, then $e^{\int f(v)dv}$ is an integrating factor, which is a function of x+y.

(iv) If $\dfrac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q\,y - P\,x} = f(xy) = f(v)$, then $e^{\int f(v)dv}$ is an integrating factor, which is a function of xy.

[Example] $(4x + 3y^2)\,dx + 2xy\,dy = 0$

$P = 4x + 3y^2 \quad Q = 2xy$

$$\frac{\partial P}{\partial y} = 6y \neq 2y = \frac{\partial Q}{\partial x} \quad \text{—— not exact}$$

Check $$\frac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q} = \frac{6y - 2y}{2xy} = \frac{2}{x} = f(x)$$

Thus, the integrating factor F(x) is

$$F(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = x^2$$

Multiply F(x) on both sides of the differential equation, we have

$$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 0 \quad \text{—— Exact}$$

$$\Rightarrow \quad x^4 + x^3y^2 = c$$ #

[Example] $2 \cosh x \cos y \, dx - \sinh x \sin y \, dy = 0$

Note that $\cosh x \equiv \dfrac{e^x + e^{-x}}{2}$

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}$$

and $d \cosh x / dx = \sinh x$

$d \sinh x / dx = \cosh x$

$P = 2 \cosh x \cos y$

$Q = - \sinh x \sin y$

$$\frac{\partial P}{\partial y} = -2 \cosh x \sin y \neq -\cosh x \sin y = \frac{\partial Q}{\partial x}$$

Check $$\frac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q} = \frac{-2 \cosh x \sin y + \cosh x \sin y}{-\sinh x \sin y}$$

$$= \frac{\cosh x}{\sinh x} = f(x)$$

Thus, the integrating factor F(x) is

$$F(x) = e^{\int f(x)dx} = e^{\int \frac{\cosh x}{\sinh x} dx} = \sinh x$$

Multiply sinh x on both sides of the differential equation, we have

$$2 \sinh x \cosh x \cos y \, dx - \sinh^2 x \sin y \, dy = 0$$

which can be solved by

$$u = \int FP dx + k(y) = \int 2 \sinh x \cosh x \cos y \; dx + k(y)$$

$$= \sinh^2 x \cos y + k(y)$$

and $\quad \frac{\partial u}{\partial y} = F\,Q \quad \Rightarrow - \sinh^2 x \sin y + k'(y) = - \sinh^2 x \sin y$

$\Rightarrow \quad k' = 0 \quad \rightarrow \quad k(y) = \text{constant}$

Thus the solution to the above ODE is: $\quad \sinh^2 x \cos y = c$ #

[Exercise] $x\,y\,dx + (x^2 + y^2 + 1)\,dy = 0$

[Exercise] $(y^2 + x\,y + 1)\,dx + (x^2 + x\,y + 1)\,dy = 0$

[Question] Is the integrating factor of a given ODE unique? **NO!**

$$ydx + 2xdy = 0$$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 2$$

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q} = -\frac{1}{2x}$$

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{P} = -\frac{1}{y}$$

7 Linear Differential Equations

7.1 Definitions

Linear Differential Equations

An n^{th}–order differential equation is *linear* if it can be written in the form

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_o(x)\,y = f(x)$$

Hence, a first-order linear equation has the form

$$\frac{dy}{dx} + p(x)\,y = r(x)$$

e.g.,

$y' - y = e^{2x}$ 1^{st}– order linear

$y' - \dfrac{y}{x} = -\dfrac{5}{2}x^2 y^3$ 1^{st}–order nonlinear

$y'' + a(x)\,y' + b(x)\,y = f(x)$ 2^{nd}–order linear

Homogeneous Differential Equations

If the function f(x) = 0 [or r(x) = 0], then the above linear differential equation is said to be *homogeneous*; otherwise, it is said to be *nonhomogeneous*.

e.g. $y' - y = 0$ homogeneous

$y' - y = e^{2x}$ nonhomogeneous

7.2 Solution of the First-Order Linear Differential Equations

Homogeneous Equation

The solution of the linear homogeneous equation

$$y' + p(x)\, y = 0$$

can be obtained by *separation of variables*

$$\frac{dy}{y} = -\, p(x)\, dx$$

or $$y(x) = c\, e^{-\int p(x)\, dx}$$

Nonhomogeneous Equations

The nonhomogeneous equation

$$y' + p(x)\,y \;=\; r(x)$$

can be written in the following form

$$[p(x)\,y - r(x)]\,dx + dy \;=\; 0$$

which is of the form

$$P(x)\,dx + Q(x)\,dy = 0$$

with $\quad P(x) = p(x)\,y - r(x)$

$$Q(x) = 1$$

Since $\quad \dfrac{\partial P(x)}{\partial y} = p(x) \neq 0 = \dfrac{\partial Q}{\partial x}$

the above equation is not exact differential.

However,

$$\frac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q} = p(x)$$

we have the integrating factor

$$F(x) = e^{\int P(x)dx}$$

for the differential equation. Multiply the differential equation by the integrating factor, we have

$$[y' + p(x)\, y]\, e^{\int p(x)\, dx} = y'e^{\int p(x)dx} + yp(x)e^{\int p(x)dx} = r(x)\, e^{\int p(x)\, dx}$$

According to chain rule, the left side of the above equation is the derivative of $ye^{\int p(x)\,dx}$, i.e., $\frac{d}{dx}\left[y\, e^{\int p(x)\,dx} \right] = r(x)\, e^{\int p(x)\,dx}$

Integrating both sides of the above equation wrt x, we have

$$y\, e^{\int p(x)\,dx} = \int r(x)\, e^{\int p(x)\,dx}\, dx) \;\; + c$$

or $$y = e^{-\int p(x)\,dx} \left[\int r(x)\, e^{\int p(x)\,dx}\, dx \;\; + c \right]$$

Alternative Solution Procedure：

i. Rearrange the equation in the ***standard form***:

$$y' + p(x)\, y = r(x)$$

ii. Derive the integrating factor: $e^{\int p(x)\, dx}$

iii. Multiply both sides of the given equation by this factor

iv. Integrate both sides of the resulting equation. ***Note that the integral of the left is always just y times the integrating factor.***

v. Solve the integrated equation for y.

[Example] $y' = y + x^2, \quad y(0) = 1$

$$y' - y = x^2 \quad \Rightarrow \quad y' + p(x)\,y = r(x)$$

thus, the integrating factor is: $e^{\int p(x)\,dx} = e^{\int -1\,dx} = e^{-x}$

Multiply both sides of the differential equation according to the alternative procedure, we have

$$e^{-x}(y' - y) = x^2 e^{-x} \qquad \text{or} \qquad (y\,e^{-x})' = x^2 e^{-x}$$

Integrating both sides, we have:

$$y\,e^{-x} = \int x^2 e^{-x}\,dx + c = c - (x^2 + 2x + 2)\,e^{-x}$$

Thus, $y = c\,e^{x} - (x^2 + 2x + 2)$ —— general solution

Since $y(0) = 1 \quad \Rightarrow \quad c = 3 \quad \therefore y = 3\,e^{x} - (x^2 + 2x + 2)$
——particular solution

The general solution can also be obtained by the general formula

$$y = e^{-\int p(x)\,dx}\left[\int r(x)\, e^{\int p(x)\,dx}\,dx + c\right]$$

$$= e^{x}\left[\int x^2 e^{-x}\,dx + c\right] = c\,e^{x} - (x^2 + 2x + 2)$$

[Example] $\dfrac{dy}{dx} = x^3 - 2xy \qquad y(1) = 1$

[Solution] $y' + 2xy = x^3 \quad \Leftrightarrow \quad y' + p(x)y = r(x)$

$\therefore$ The integrating factor is: $e^{\int p(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2}(y' + 2xy) = x^3 e^{x^2}$ or $(e^{x^2} y)' = x^3 e^{x^2}$

$$e^{x^2} y = \int x^3 e^{x^2}\,dx + c$$

$\therefore \; y = e^{-x^2}\left[\int x^3 e^{x^2}\,dx + c\right]$ (integration by parts)

$$= e^{-x^2}\left[e^{x^2}\left(\frac{x^2-1}{2}\right) + c\right] = \frac{x^2-1}{2} + c\,e^{-x^2}$$

Since $y(1) = 1 = \dfrac{1-1}{2} + ce^{-1}$, $c = e$. Thus, $\boxed{y = \dfrac{x^2-1}{2} + e^{1-x^2}}$ #

Bernoulli's Equations

The equation

$$\frac{dy}{dx} + p(x)\, y = g(x)\, y^{a} \qquad (\text{ a is any real number })$$

which is known as the *Bernoulli's Equation*, can be reduced to linear form by a suitable change of the dependent variables. For a = 0 and a = 1, the equation is linear, and otherwise, **it is nonlinear**. Set

$$u(x) = y^{1-a}$$

then $$u'(x) = \boxed{(1-a)\, y^{-a}}\, y'$$

so if we multiply both sides of the differential equation by $(1-a)\, y^{-a}$, we obtain

$$(1-a)\, y^{-a}\, y' + (1-a)\, p(x)\, y^{1-a} = (1-a)\, g(x)$$

or $$u' + (1-a)\, p(x)\, u = (1-a)\, g(x)$$

The equation is now *linear* and may be solved as before.

[Example] $y' - \dfrac{y}{x} = -\dfrac{5}{2} x^2 y^3$

[Solution] Compare the above equation with $y' + p(x)\, y = g(x)\, y^a$

we have $a = 3$ $\therefore$ Set $u(x) = y^{1-a} = y^{-2}$ $\Rightarrow$ $u'(x) = -2\, y^{-3}\, y'$

Multiply both sides of the equation by $-2\, y^{-3}$, we obtain

$-2\, y^{-3}\, y' + \dfrac{1}{x} 2\, y^{-2} = 5\, x^2$ or $u' + \dfrac{2}{x} u = 5\, x^2$ (which is a first order linear ODE).

The integrating factor is then: $e^{\int \frac{2}{x} dx} = x^2$

multiply both sides of the differential equation of u by x^2, we have

$$x^2 u' + 2\, x\, u = 5\, x^4 \quad \text{or} \quad (x^2 u)' = 5\, x^4$$

$\Rightarrow x^2 u = x^5 + c$ or $x^2 y^{-2} = x^5 + c \Rightarrow$ $y = \pm\, (x^3 + c\, x^{-2})^{-1/2}$ #

[Exercise] Show that the differential equation

$$y' + p(x)\, y = f(x)\, y \ln y$$

can be made linear if we set $u = \ln y$

Riccati's Equation

$$y' + g(x)\, y + h(x)\, y^2 = k(x)$$

- Except in special instances, the solution cannot be given in closed form
- If one particular solution is known, then the remaining solutions can be explicitly derived.

Riccati's Equation

Consider two distinct solutions

$$y(x) \neq \phi(x) \quad \text{(given)}$$

Let

$$u(x) = y(x) - \phi(x)$$

$$\therefore u' + gu + h\left(y^2 - \phi^2\right) = 0$$

$$\text{since } y^2 - \phi^2 = (y-\phi)(y+\phi) = (y-\phi)(y-\phi+2\phi) = u(u+2\phi)$$

$$\Rightarrow \text{Bernoulli's Equation: } u' + [g + 2\phi h]u + hu^2 = 0$$

Let $\quad v = u^{1-2} = u^{-1} \qquad \therefore v' = -u^{-2}u'$

$$-u^{-2}u' - u^{-2}[g + 2\phi h]u = h \ \Rightarrow \text{1st-order linear ODE: } v' - [g + 2\phi h]v = h$$

$$\therefore y(x) = \phi(x) + u(x) = \phi(x) + \frac{1}{v(x)}$$

[Example] $y' - \frac{1}{x} y - \frac{1}{x} y^2 = -\frac{2}{x}$ and $\phi(x) = 1$

$$\therefore y(x) = 1 + \frac{1}{v(x)} \quad \text{and} \quad y'(x) = -\frac{1}{v^2(x)} v'(x)$$

$$-\frac{1}{v^2} v' - \frac{1}{x}\left(1 + \frac{1}{v}\right) - \frac{1}{x}\left(1 + \frac{1}{v}\right)^2 = -\frac{2}{x} \quad \Rightarrow \quad v' + \frac{3}{x} v = -\frac{1}{x}$$

Integrating Factor $e^{\int \frac{3}{x} dx} = x^3$

$$x^3 v' + x^3 \frac{3}{x} v = -x^3 \frac{1}{x} \Rightarrow \left(x^3 v\right)' = x^3 v' + 3x^2 v = -x^2$$

$$\Rightarrow x^3 v = -\frac{1}{3} x^3 + c \Rightarrow v = -\frac{1}{3} + \frac{c}{x^3}$$

$$\therefore y(x) = 1 + \frac{1}{v(x)} = 1 + \frac{1}{-\frac{1}{3} + \frac{c}{x^3}} = \frac{k + 2x^3}{k - x^3} \qquad \text{where } k = 3c$$

[Exercise] $y' - 2xy - y^2 = 2$ and $\phi(x) = -\dfrac{1}{x}$

Thus, we have $g(x) = -2x,\ h(x) = -1,\ k(x) = 2,\ y(x) = -\dfrac{1}{x} + \dfrac{1}{v(x)}$

$$v' - (g + 2\phi h)v = h \ \Rightarrow\ v' - \left[-2x + 2\left(-\frac{1}{x}\right)(-1)\right]v = -1 \ \Rightarrow\ v' + \left(2x - \frac{2}{x}\right)v = -1$$

$$F(x) = e^{\int\left(2x - \frac{2}{x}\right)dx} = e^{x^2 - 2\ln x} = \frac{e^{x^2}}{x^2} \ \Rightarrow\ \frac{d}{dx}\left(v\frac{e^{x^2}}{x^2}\right) = -\frac{e^{x^2}}{x^2}$$

$$\therefore v\frac{e^{x^2}}{x^2} = -\int\frac{e^{x^2}}{x^2}dx + c = -\left(-\frac{1}{x}e^{x^2} + 2\int e^{x^2}dx\right) + c = \left[\frac{1}{x}e^{x^2} - eE(x)\right] + c$$

$$v(x) = \left[x - 2x^2e^{-x^2}E(x)\right] + cx^2e^{-x^2}$$

$$\Rightarrow y(x) = -\frac{1}{x} + \frac{1}{x + \left[c - 2E(x)\right]x^2e^{-x^2}}$$

8 Applications of First-Order Differential Equations - Modeling

[Example 1]

A tank is initially filled with 100 gal of salt solution containing 0.5 lb of salt per gallon. Fresh brine containing 3 lb of salt per gallon runs into the tank at the rate of 2 gal/min, and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t.

Let Q lb be the total amount of salt in solution in the tank at any time t, and let dQ be the increase in this amount during the infinitesimal interval of time dt. At any time t, the amount of salt per gallon of solution is therefore Q/100 (lb/gal). The material balance of salt in the tank is

$$\left\{\begin{matrix}\text{Rate of Accum.}\\ \text{of Salt}\\ \text{in the Tank}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate of Salt}\\ \text{Flow}\\ \text{into the Tank}\end{matrix}\right\} - \left\{\begin{matrix}\text{Rate of Salt}\\ \text{Flow}\\ \text{out of the Tank}\end{matrix}\right\}$$

The rate at which salt enters the tank is

$$2 \text{ gal/min} \times 3 \text{ lb/gal} = 6 \text{ lb/min}$$

Likewise, since the concentration of slat in the mixture as it leaves the tank is the same, as the concentration Q/100 in the tank itself, the rate of salt leaves the tank is

$$2 \text{ gal/min} \times \frac{Q}{100} \text{ lb/gal} = \frac{Q}{50} \text{ lb/min}$$

Hence, the rate of accumulation of salt in the tank dQ/dt is

$$\frac{dQ}{dt} = 6 - \frac{Q}{50}$$

This equation can be written in the form

$$\frac{dQ}{300 - Q} = \frac{dt}{50}$$

and solved as a *separable* equation, or it can be written

$$\frac{dQ}{dt} + \frac{Q}{50} = 6$$

and treated as *a linear equation*.

Considering it as a linear equation, we must first compute the integrating factor $e^{\int p(x)\, dt}$ =

$$e^{\int \frac{1}{50}\, dt} = e^{t/50}$$

Multiplying the differential equation by this factor gives

$$e^{t/50}\left[\frac{dQ}{dt} + \frac{Q}{50}\right] = 6\, e^{t/50}$$

From this, by integration, we obtain

$$Q\, e^{t/50} = 300\, e^{t/50} + c$$

or $$Q = 300 + c\, e^{-t/50}$$

Substituting the initial conditions $t = 0$, $Q = 50$, we find $c = -250$

Hence, $$Q = 300 - 250\, e^{-t/50} \quad \#$$

[Example 2]

A tank is initially filled with 100 gal of salt solution containing 0.5 lb of salt per gallon. Fresh brine containing 1 lb of salt per gallon runs into the tank at the rate of 3 gal/min, and the mixture, assumed to be kept uniform by stirring, runs out at the 2 gal/min. Find the amount of salt in the tank at any time t.

In this case, the rate at which salt enters the tank is

$$3 \text{ gal/min} \times 1 \text{ lb/gal} = 3 \text{ lb/min}$$

Since the amount of brine in the tank increases with time (at 1 gal/min), the concentration of salt in the tank is then

$$\frac{Q}{100 + t} \text{ lb/gal}$$

Therefore, the rate of salt leaves the tank is

$$2 \text{ gal/min} \times \frac{Q}{100 + t} \text{ lb/gal} = \frac{2\,Q}{100 + t} \text{ lb/min}$$

From the mass balance of salt of the system, we have

$$\frac{dQ}{dt} = 3 - \frac{2Q}{100 + t}$$

or
$$\frac{dQ}{dt} + \frac{2\,Q}{100 + t} = 3$$

The integrating factor in this case is

$$e^{\int p(x)\,dt} = e^{\int \frac{2}{100 + t}\,dt} = (100 + t\,)^2$$

So that, we have

$$[\,(\,100 + t\,)^2\,Q\,]' = 3\,(\,100 + t\,)^2$$

or
$$Q(t) = (\,100 + t\,) + c\,(\,100 + t\,)^{-2}$$

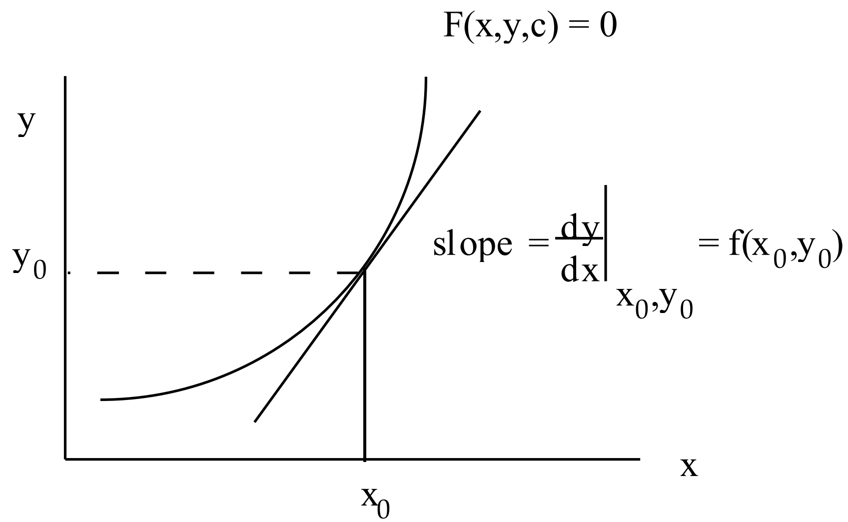
Setting t = 0, we find that $c = -50(100)^2$, so that

$$Q(t) = 100 + t - 50\left[\,1 + \frac{t}{100}\,\right]^{-2} \quad \#$$

9 Approximate Solutions

9.1 Method of Direction Fields — Graphic Method

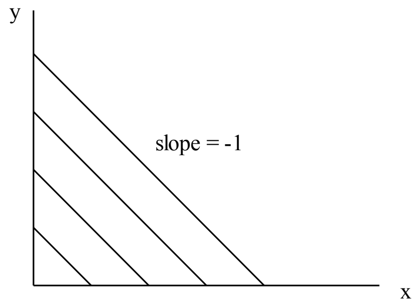
Slope of a Curve



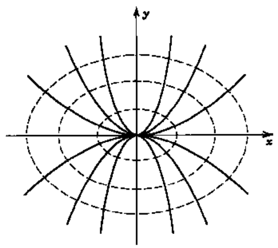
$$y' = f(x,y) \quad \Rightarrow \quad y = g(x\,;c)$$

$$\text{or} \quad F(x,y,c) = 0$$

[Example] $y' = -1 \Rightarrow \quad y = -x + c \quad$ or $\quad x + y = c$

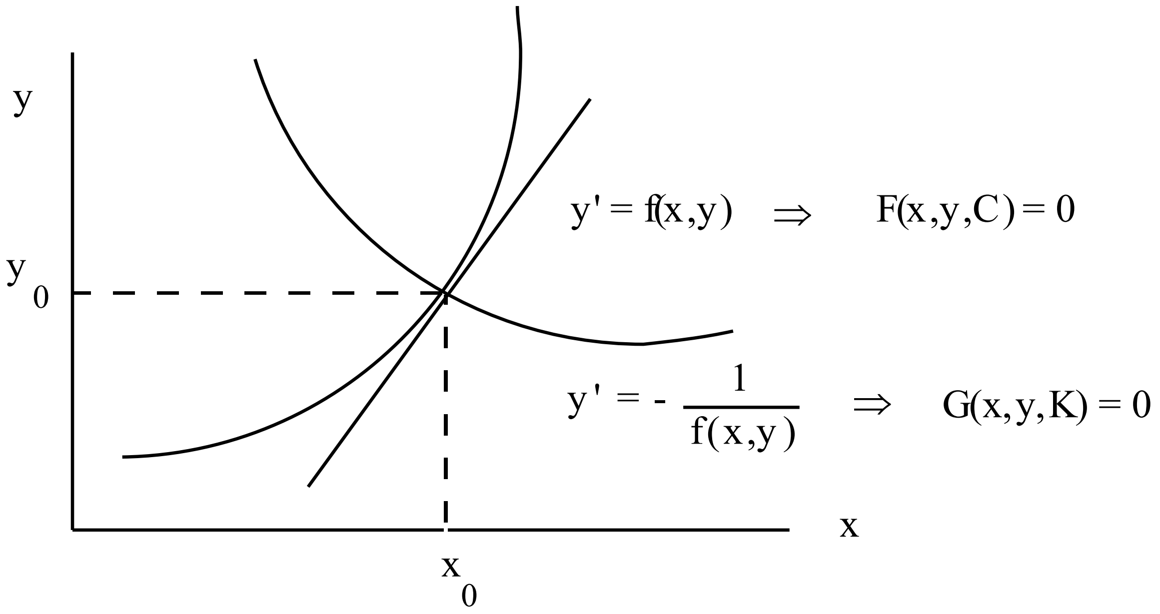


[Example] $y' = \dfrac{2y}{x} \quad \Rightarrow \quad y = c\,x^2$



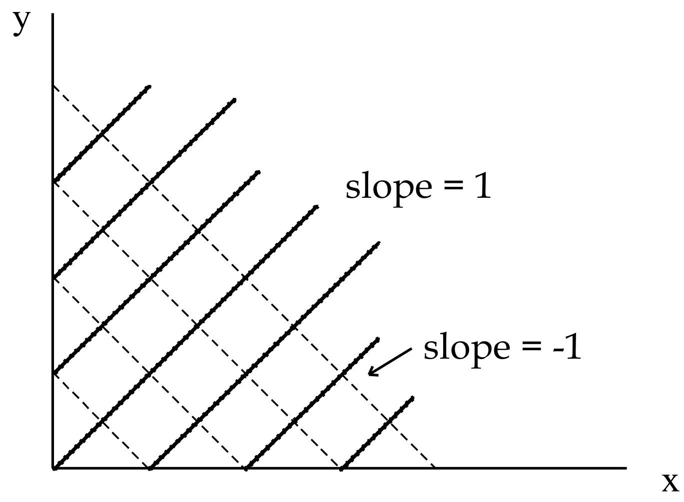
Orthogonal Trajectories

The curves of a family C is said to be *orthogonal trajectories* of the curves of a family K, and vice versa, if at every intersection of a curve of C with a curve of K, the two curves are *perpendicular*.



[Example] $y' = -1 \Rightarrow \quad y = -x + C$ or $x + y = C$

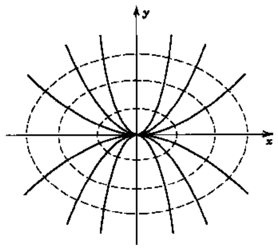
$y' = 1 \quad \Rightarrow \quad y = x + K$ or $y - x = K$



The two families of curves, $x + y = C$ & $y - x = K$, are *orthogonal trajectories* of each other.

[Example] $$y' = \frac{2\,y}{x}$$

$\Rightarrow$ $y = C\,x^2$ (the Solid Lines in the following Figure)



Orthogonal trajectories: (the Dashed Lines in the above Figure)

$$y' = -\frac{1}{\frac{2\,y}{x}} = -\frac{x}{2\,y} \quad \Rightarrow \quad \frac{x^2}{2} + y^2 = K$$

Method of Direction Fields for $y' = f(x,y)$

Step 1 Plot the isoclines (curves of constant slope) of $y' = f(x,y)$, i.e., plot the curves for

$$f(x,y) = k = \text{constant}$$

Step 2 Draw a number of parallel short line segments (lineal elements) with slope k along each isocline $f(x,y) = k$

Step 3 Connect the lineal elements to get the approximate solution curves.

[Example] $y' = x\,y$ or $\dfrac{dy}{y} = x\,dx$

$\Rightarrow$ $\ln y = \dfrac{1}{2}x^2 + c$ or $y = c'\, e^{x^2/2}$

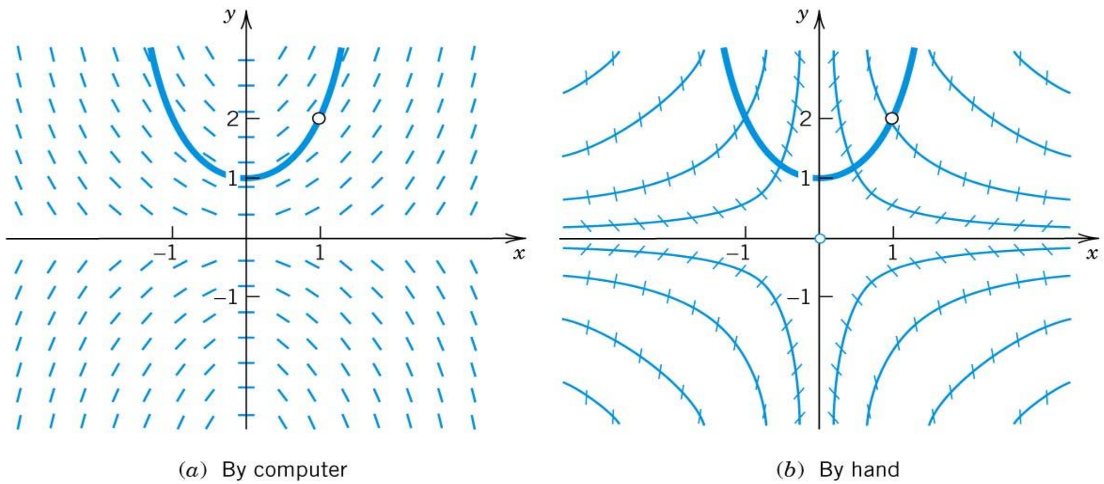


Fig. 5. Direction field of $y' = xy$

9.2 Picard's Iteration Method –– Successive Approximations

Iteration Method :

Assume we need to solve the (positive) value of x of the algebraic equation:

$$x^2 + x = 1$$

or, alternatively

$$x = \sqrt{1 - x}$$

Since the above equation is nonlinear, we propose to solve the value of x by iteration if we assume that the initial guess of x be 0.5, i.e.,

$$x_o = 0.5$$

then the 1st approximation of x, $x^{(1)}$, can be calculated by

$$x^{(1)} = \sqrt{1 - x_o} = \sqrt{1 - 0.5} = 0.707$$

Similarly, the 2nd approximation of x, $x^{(2)}$, is then

$$x^{(2)} = \sqrt{1 - x^{(1)}} = \sqrt{1 - 0.707} = 0.541$$

By the same token, the $(n+1)^{th}$ approximation of x can be solved by

$$x^{(n+1)} = \sqrt{1 - x^{(n)}}$$

where $x^{(n)}$ is the n^{th} approximation of x. We then obtain, successively, that

$$\Rightarrow \qquad \ldots \quad x^{(5)} = 0.657 \quad \ldots \qquad x^{(\infty)} = 0.618$$

Similarly, consider the initial value problem

$$y' = f(x, y) \quad ; \qquad y(x_0) = y_0$$

Integrate both sides of the differential equation from x_0 to x with respect to x yields

$$\int_{x_0}^{x} y' \, dx = \int_{x_0}^{x} f(x,y(x)) \, dx$$

or

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t,y(t)) \, dt$$

Since the function y(t) in the integrand is not known *a priori*, the integral of the right-hand-side of the above equation can not evaluated unless the approximations of y(t) are introduced. We now define a sequence of functions $\{y_n(x)\}$, called Picard iterations, by the successive formulas

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t,y_0(t)) \, dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t,y_1(t))\,dt$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t,y_2(t))\,dt$$

.

.

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t,y_{n-1}(t))\,dt$$

Remarks: Picard's method is of great theoretical values in connection with Picard's existence and uniqueness theorem. Its practical value is limited because it involves integrations that may be complicated.

[Example] Consider the initial value problem

$$y'(x) = y \quad ; \quad y(0) = 1$$

In this case, $f(x,y) = y(x)$

$$y_0(x) = y_0 = 1$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t,y_0(t))\,dt = 1 + \int_{0}^{x} (1)\,dt = 1 + x$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt = 1 + \int_0^x (1 + t)\, dt$$

$$= 1 + x + \frac{x^2}{2}$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))\, dt$$

$$= 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Finally, we will have

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Hence, $$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

which converges to the exact solution e^x.

10 Existence and Uniqueness of IVP Solutions

[Examples]

(i) $|y'| + |y| = 0$, $y(0) = 0 \Rightarrow$ Trivial solution, $y = 0$;
$y(0) = 1 \Rightarrow$ No solution exists.

(ii) $y' = x$, $y(0) = 1 \Rightarrow$ unique solution, $y = x^2/2 + 1$.

(iii) $x\,y' = y - 1$, $y(0) = 1$

$\Rightarrow$ infinitely many solutions, $y = 1 + c\,x$.

Questions: **Is there a solution to the problem?**

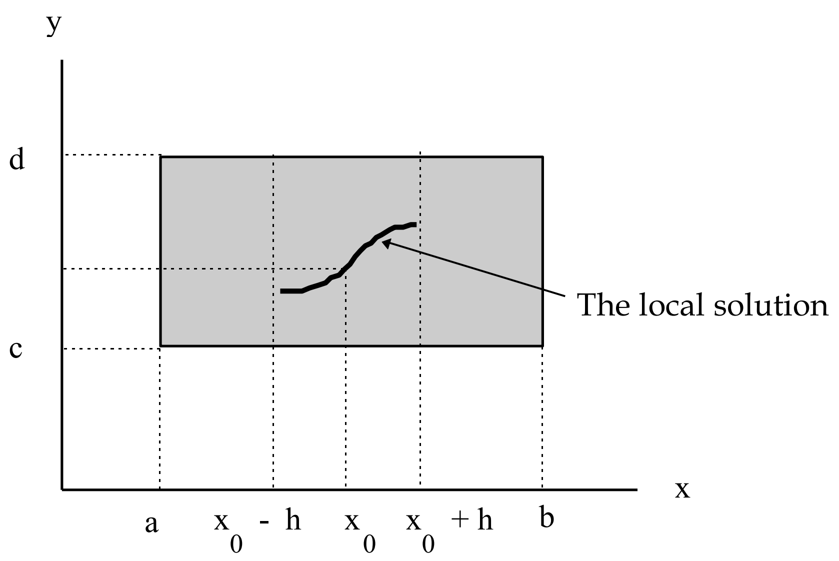
Is the solution unique?

Existence–Uniqueness Theorem:

If f and $\frac{\partial f}{\partial y}$ are ***continuous*** and ***bounded*** in a rectangle R given by a < x < b, c < y < d that contains the point (x_0,y_0) (see the following figure),

then, in an interval $x_0 - h < x < x_0 + h$ contained in a < x < b, there is a unique solution y = y(x) of the initial value problem

$$\frac{dy}{dx} = f(x,y) \quad ; \qquad y(x_0) = y_0$$



Note that the above theorem **is valid for a small region around the initial point,** we call such a theorem a *local* existence–uniqueness theorem.

[Example] Check the initial value problem

$$\frac{dy}{dx} = x^2 + y^3 \qquad y(0) = 1$$

Since $f(x,y) = x^2 + y^3$ and $\frac{\partial f}{\partial y} = 3\,y^2$ are continuous everywhere, they are continuous in any rectangle R containing the initial point (0,1). Hence, a unique local solution exists.

[Example] Check the initial value problem

$$\frac{dy}{dx} = x\,y^{1/3} y(0) = 0$$

Since $\frac{\partial f}{\partial y} = \frac{x}{3\,y^{2/3}}$ is **<u>not bounded</u>** at the initial point (0,0), the above theorem does not apply to this problem. Indeed, the problem has two solutions

$$y = 0 \text{ and } y = \frac{x^3}{3\sqrt{3}}$$

[Exercise] Is the solution $y = x^2/4$ to the differential equation $y' = \sqrt{y}$ with the initial condition $y(0) = 0$ unique?

11 Review Questions and Problems of Chapter 1

(I) Solve the following Differential Equations:

1. $y' + 2xy = -6x$
2. $y\,dx - dy = x^2 y^2\,dx + x\,dy$
3. $(x^2 + y^2)\,y' + (2xy + 1) = 0,\ y(2) = -2$
4. $2xy' = 10x^3 y^5 + y$

5 $\quad y' = \left[\dfrac{2x + y - 1}{x - 2}\right]^2$

6 $\quad (4xy + 6y^2) + (2x^2 + 6xy)\,y' = 0$

7 $\quad x^2 y' - 3xy = -2y^{5/3}$

8 $\quad (4y^3 - x)\dfrac{dy}{dx} = y$

9 $\quad xy' - 2y = xe^x$

10 $\quad 2xyy' + (x - 1)\,y^2 = x^2 e^x$

11 $\quad y' = xe^{-x^3} - 3x^2 y \qquad$ with $y(0) = -1$

12 $\quad (x^2 + y^2 + 1)\,y' + xy = 0$

13 $\quad (y + \tan(x + y))\,y' + y = 0$

14 $\quad x^2 y' = y^2 + 5xy + 4x^2$

15 $\quad (3xe^y + 2y)\,dx + (x^2 e^y + x)\,dy = 0$

16 $(x + y - 2)\,y' + (x - y) = 0$

17 $(1 + x^2)\,dy + x\,y\,dx = \sqrt{1+x^2}\;dx$

18 $(2x + 3y - 5)\,y' + (x + 2y - 3) = 0$

19 $y' = \dfrac{y\,e^x}{e^x + (y+1)\,e^y}$

20 $y' = \dfrac{2\,x\,e^x - y^2}{2\,y}$ with $y(0) = \sqrt{2}$

21 $y' = \dfrac{x - y^2}{y}$

II. Under what conditions is the following differential equation exact?

$$(c\,x^2 y\,e^y + 2\cos y) + (x^3 e^y\,y + x^3 e^y + k\,x \sin y)\,y' = 0$$

Solve the exact equation.

III. Apply Picard's iteration method to the following initial value problems.

(i) $y' - x\,y = 1$, ; $y(0) = 1$

(ii) $y' = x\,y$; $y(0) = 1$

IV Find the orthogonal trajectories of the circles.

$$x^2 + (y - c)^2 = 1 + c^2$$

Summary

1. Separation of Variables

$$g(y)\,dy = f(x)\,dx$$

$$\int g(y)\;dy = \int f(x)\;dx + c$$

(a) $y' = f(ax + by)$ where a and b are constants

Let $u = ax + by$

$$\frac{du}{dx} = a + b\frac{dy}{dx} = a + b\,f(u)$$

$$\Rightarrow \quad \int \frac{du}{a + b\,f(u)} = \int dx + c$$

(b) $y' = f(y/x)$

Let $y/x = u$ or $y = x\,u$

(c) $y' = f\left(\dfrac{Ax + By + C}{ax + by + c}\right)$

2. Exact Differential Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Check $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Find u such that

$$\frac{\partial u}{\partial x} = M(x,y) \qquad \frac{\partial u}{\partial y} = N(x,y)$$

Then the general solution is

$$u(x,y) = c$$

Integrating Factor:

(a) If $\dfrac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q} = f(x)$, then $e^{\int f(x)dx}$ is an integrating factor.

(b) If $\dfrac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{P} = f(y)$, then $e^{-\int f(y)dy}$ is an integrating factor.

(iii) If $\dfrac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q - P} = f(x+y) = f(v)$, then $e^{\int f(v)dv}$ is an integrating factor.

(iv) If $$\frac{\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right]}{Q\,y - P\,x} = f(xy) = f(v),$$ then $e^{\int f(v)dv}$ is an integrating factor.

3. Linear Differential Equations

$$\frac{dy}{dx} + p(x)\,y = r(x)$$

Solution Procedure :

i. Write the equation in the standard form: $y' + p(x)\,y = r(x)$

ii. Compute the integrating factor $e^{\int p(x)\,dx}$

iii. Multiply both sides of the given equation by this factor

iv. Integrate both sides of the resulting equation. ***<u>Note that the integral of the left is always just y times the integrating factor</u>.***

v. Solve the integrated equation for y.

<u>Bernoulli's Equations</u>

$$\frac{dy}{dx} + p(x)\,y = g(x)\,y^a \qquad (a \neq 0 \text{ or } 1)$$

Set

$$u(x) = y^{1-a} \quad \Rightarrow \quad u' + (1-a)\,p(x)\,u = (1-a)\,g(x)$$